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Holographic Superfluids and Solitons

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ACADEMIC DISSERTATION

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Abstract

Superfluidity is perhaps one of the most remarkable observed macroscopic quantum effect. It appears when a macroscopic number of particles occupies a single quantum state. Using modern experimental techniques one can manipulate the wavefunction of the superfluid to create coherent structures such as domain walls (often called dark solitons) and vortices. There is a large literature on theoretical work studying the properties of such solitons using semiclassical methods.

This thesis describes an alternative method for the study of superfluid solitons. The method used here is a holographic duality between a class of quantum field theories and gravitational theories. The classical limit of the gravitational system maps into a strong coupling limit of the quantum field theory. We use a holographic model of superfluidity to study solitons in these systems. One particularly appealing feature of this technique is that it allows us to take into account finite temperature effects in a large range of temperatures.

List of Publications

The content of this thesis is based on the following research articles:

I. V. Keranen, E. Keski-Vakkuri, S. Nowling, K. P. Yogendran, "Dark Solitons in Holographic Superfluids," Phys. Rev. **D80** (2009) R121901. [arXiv:0906.5217 [hep-th]].

II. V. Keranen, E. Keski-Vakkuri, S. Nowling, K. P. Yogendran, "Inhomogeneous Structures in Holographic Superfluids: I. Dark Solitons," Phys. Rev. **D81** (2010) 126011. [arXiv:0911.1866 [hep-th]].

III. V. Keranen, E. Keski-Vakkuri, S. Nowling, K. P. Yogendran, "Inhomogeneous Structures in Holographic Superfluids: II. Vortices," Phys. Rev. **D81** (2010) 126012. [arXiv:0912.4280 [hep-th]].

IV. V. Keranen, E. Keski-Vakkuri, S. Nowling, K. P. Yogendran, "Solitons as Probes of the Structure of Holographic Superfluids," New J. Phys. **13** (2011) 065003. [arXiv:1012.0190 [hep-th]].

In all of the papers the authors are listed alphabetically according to the particle physics convention.

The Author's Contribution to the Joint Publications

The author came up with the idea of studying solitons in the holographic superfluid models.

In the first paper, the Mathematica code was developed by the current author, S. Nowling and K. P. Yogendran. The author came up with the idea of studying the density depletion fraction and on comparing the density depletions with those of solitons in the BCS-BEC crossover. The results were analyzed and the paper was written jointly among the collaborators.

On the second paper, the author came up with the way of performing the error analysis. The results where analyzed and the paper was written jointly with all of the collaborators.

On the third paper, the author came up with the quantities to calculate, organized the calculations, and developed the code. Also the analytic expression for the free energy was derived by the author and the draft was written by the author, and later polished jointly with all of the collaborators.

On the fourth paper, the author developed the Mathematica code and organized the calculations. The results were analyzed and the paper was written jointly with all of the collaborators.

Contents

Chapter 1

Introduction

Holography provides a highly non-trivial connection between theories of quantum gravity in asymptotically anti-de Sitter ($AAdS$) spacetimes and conventional quantum field theories [1]. One way of viewing holography is that it gives a non-perturbative definition of quantum gravity in terms of a quantum field theory. It is still unclear which quantum field theories have gravitational duals and which do not. Thus, it is important to gain more understanding of what kinds of quantum field theory phenomena can be seen in gravitational theories and *vice versa*. One way of approaching the problem is to use weakly coupled semi-classical gravitational theories and study the range of quantum field theory phenomena that can be seen in such a description.

As a more practical motivation to the work we note that holography gives a new method for studying strongly coupled quantum field theories. This way it can provide us with new understanding of possible quantum field theory phenomena. Such phenomena have applications wherever quantum field theory can be used as a framework to describe a physical system. One such place is condensed matter and atomic physics, where one can find many interesting phenomena that are still not fully understood in terms of conventional quantum field theory methods. A lot of recent work in holography has been in attempting to apply it to understand high temperature superconductivity [2, 3] and non-Fermi liquids [4]. Another interesting direction of applications is the unitary regime of fermion gases in the BCS-BEC crossover [5]. So far there has been less work in this direction.

It is difficult to study solitons in interacting quantum field theories at finite temperature. Holography provides an alternative tool for such studies. From studying the solitons one can also learn more about the holographic superfluids, and possibly on the mechanism of symmetry breaking in these models. Indeed by studying the solitons we have found hints of a crossover similar to that of the BCS-BEC crossover as certain parameters of the gravitational theory are varied [6]. Currently the nature of this possible crossover is still unclear.

I have attempted to make the thesis a logical path from some simple well known aspects of superfluids to the holographic models. The first section starts with some well known features of superfluids, such as their ability to sustain frictionless and irrotational flow, to arrive at the picture of a superfluid as a quantum system which has undergone spontaneous symmetry breaking of a certain kind of a symmetry. The second section presents two example theories of superfluids and introduces the main ideas of the two fluid model of Landau and Tisza [7, 8]. The third section introduces the main ideas behind holography, and describes the best understood example of the duality, which is that between $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and type IIB superstring theory in $AdS_5 \times S^5$. Using this example and more general arguments we show how global symmetries in the dual quantum field theory are realized in the gravitational theory. This is important to model superfluidity which is identified as spontaneous symmetry breaking of a global symmetry. With all these ingredients we construct the simplest possible gravitational theory that can describe superfluidity holographically. Finally we study some basic features of this theory and show how superfluidity arises in this model. The main subject, solitons, is not touched in the introduction part as the articles are fairly self contained with that respect.

Chapter 2

Superfluidity

A superfluid is a fluid which can exhibit frictionless flow. One can derive a simple criterion for superfluidity [7] based on a quantum mechanical consideration. Before we can derive the criterion, we need to introduce some effective field theory concepts.

2.1 On effective field theory

A system made of local quantum mechanical degrees of freedom can be described in terms of a quantum field theory. If one knows that the system is made of particles with known properties, one can introduce quantum fields Φ_i to create those particles. If the effects of interactions between the quantum fields Φ_i are small enough, the system can be well described by weakly interacting particles and corrections induced by the interactions can be calculated using semiclassical methods.

If the interactions between the fields are too large, the above procedure can break down. Sometimes one can still identify the lowest lying excited states above the ground state of the system, for example by using symmetry arguments. An excited state in a quantum field theory carries a 4-momentum[1] quantum number k^μ and internal quantum numbers σ. The quantum numbers of such a state are those of a particle with internal degrees of freedom σ. Thus, one can introduce effective quantum fields $\Phi_{eff}^{(\sigma)}$ to create the lowest lying excitations. If there is a separation of energy scales to higher excited states, the low energy dynamics of the system can be determined by writing down a local action for $\Phi_{eff}^{(\sigma)}$. This action is often sufficient in determining the low energy behavior of the system. Successful examples of such a procedure

[1] Assuming translational invariance of the ground state and time independence of the Hamiltonian.

include the Landau-Fermi liquids [9, 10], meson effective actions in QCD [11] and Seiberg duality [12, 13]. The description of the low lying excited states is usually referred to as effective field theory.

The effect of ignoring the higher excited states is to dress the coupling constants in the effective action for the lowest lying excited states. This follows from general principles of local quantum field theory, and is at the heart of the renormalization group [10].

In general, the excited states need not be infinitely long lived. If the lifetime of the state is longer than its inverse energy, we will call the state a *quasiparticle*.

2.2 The Landau criterion

After introducing the concept of quasiparticles we are ready to introduce a criterion for superfluidity [7]. In this section we consider relativistic fluids. Consider a superfluid flowing with a velocity $\mathbf{v}$ with respect to a container. We will start by looking at the fluid in the comoving frame. In this reference frame, the fluid is in its ground state. Dissipation of the fluid flow occurs when energy is exchanged between the fluid and the container. Let us consider a quasiparticle with energy ϵ_p and spatial momentum $\mathbf{p}$ being excited due to the interaction with the wall. From the rest frame of the container, the energy of the quasiparticle is

$$\epsilon' = \frac{\epsilon_p + \mathbf{p}\cdot\mathbf{v}}{\sqrt{1-v^2}}, \tag{2.1}$$

where we have performed a Lorentz transformation. Dissipation occurs when $\epsilon' < 0$, so that the energy of the fluid flow decreases. The energy of the excited quasiparticle (2.1) is minimized when $\mathbf{p}$ and $\mathbf{v}$ are antiparallel. This gives us the Landau critical velocity

$$v_{crit} = \min_p \frac{\epsilon_p}{|\mathbf{p}|}, \tag{2.2}$$

where the minimum is taken over all possible quasiparticle excitations. So whenever the fluid is moving with a velocity smaller than v_{crit}, no dissipation can occur simply by these kinematical reasons.

At this point it is instructive to consider a few simple examples. Consider a fluid made of particles with some mass m with finite particle number density and no interactions. The dispersion relation of a particle in the fluid is $\epsilon_p = \sqrt{\mathbf{p}^2 + m^2}$. This is not quite right since in order to excite finite momentum states of an already existing particle we should subtract out the mass of the particle to give $\epsilon_p = \sqrt{\mathbf{p}^2 + m^2} - m$. This way we see that

$$v_{crit} = \min_p(\sqrt{\mathbf{p}^2 + m^2} - m)/|\mathbf{p}| = 0. \tag{2.3}$$

Thus, a fluid made from free massive particles cannot be a superfluid. The same argument goes through in the non-relativistic limit where $v \ll 1$, so that $\sqrt{\mathbf{p}^2 + m^2} - m \approx \mathbf{p}^2/2m$, and again $v_{crit} = 0$.

Adding weak interactions between the particles causes the physical mass to be renormalized by interactions of the particle with its surrounding medium. This will change the specific value of m in (2.3) but again $v_{crit} = 0$, unless there are more dramatic many-body effects that change the nature of the low lying excited states to be different from the free massive particles.

Let us next figure out what kind of dispersion relations are allowed for a superfluid. Clearly we need at least

$$\lim_{p \to 0} \frac{\epsilon_p}{|\mathbf{p}|} \neq 0. \tag{2.4}$$

So the excitation spectrum has to be of the form $\epsilon_p \propto |\mathbf{p}|^\nu$ for $\nu \leq 1$, when $|\mathbf{p}|$ is small. When $\nu = 0$ the excitations are gapped, while $\nu = 1$ corresponds to a linear dispersion relation. Then again ν could in principle be a fractional number. We will not consider such situations further in this thesis.

To have superfluidity we need a many-body effect, that affects the dispersion relation of all the quasiparticles in a way that they all satisfy the Landau criterion. In fact this is a bit too much to require in general. If the superfluid flow carries a conserved current J_μ (we will later specify what this current might be), then it is only necessary for the degrees of freedom that carry this current to satisfy the Landau criterion.

2.3 Vorticity

A further property of a superfluid is that it exhibits stable potential flow [14]. We assume that the superfluid carries a conserved current J_μ. We interpret potential flow as meaning that the expectation value of the spatial part of the current $\langle \mathbf{J}(x) \rangle$ has a vanishing vorticity

$$\nabla \times \langle \mathbf{J}(x) \rangle = 0. \tag{2.5}$$

This is easily satisfied in a low energy effective theory if

$$\mathbf{J} = \kappa_J \nabla \phi, \tag{2.6}$$

for some constant κ_J and for some new effective field ϕ. One should note that in the identification (2.6), there is a shift symmetry $\phi \to \phi + a$ for a constant a.

We would like to write down a low energy effective action for the new field ϕ. For the effective field theory to be consistent, the mass (or energy gap) of the ϕ field should be considerably smaller than the cut-off scale of the theory.

Generically in a quantum field theory, the dimensionful coupling constants in the low energy effective action are of the order of the cut-off scale [10]. So in order to keep the ϕ excitation light, we will assume that there is a shift symmetry $\phi \to \phi + a$ that prevents the generation of a mass term. Because of the shift symmetry, the effective action can only depend on derivatives of ϕ. We will not assume Lorentz invariance from this effective theory. By locality we assume that there are no fractional powers of spatial or time derivatives. This leads to an effective action

$$S_{eff} = \int d^d x \Big(\rho \partial_t \phi + \frac{1}{2}\kappa_t (\partial_t \phi)^2 - \frac{1}{2}\kappa_x (\nabla \phi)^2 + ... \Big), \tag{2.7}$$

where the dots denote terms that are, by dimensional analysis, accompanied with negative powers of the cut-off scale, and can be ignored at low energies. The coefficients ρ, κ_t and κ_x are constants. The shift symmetry leads to a conserved current, which can be seen from the action (2.7) to be

$$J_i = \kappa_x \partial_i \phi, \quad J_0 = \kappa_t \partial_t \phi + \rho. \tag{2.8}$$

So it seems that we can identify this current with (2.6), by setting $\kappa_J = \kappa_x$. Also the ground state has a non-vanishing charge density $J_0 = \rho$.

Next we can work out the dispersion relation for the φ fluctuations

$$\epsilon_p = \sqrt{\frac{\kappa_x}{\kappa_t}} |\mathbf{p}|. \tag{2.9}$$

This dispersion relation is indeed consistent with the Landau criterion.

So we have arrived at a picture of a superfluid as a quantum mechanical system which has a shift symmetry $\phi \to \phi + a$ with a corresponding conserved current.

2.4 Spontaneous symmetry breaking

One way to obtain scalar fields with shifting symmetries is by spontaneous symmetry breaking which we will shortly review in this section. The only case we will discuss in this thesis is when the broken symmetry is Abelian. Before symmetry breaking there is a conserved charge Q that generates the symmetry transformations in the Hilbert space of the quantum system. The symmetry is linearly realized in the fields if under a symmetry transformation the field transforms as $\Phi \to e^{i\alpha q}\Phi$, where q is the charge of the field Φ.

The symmetry is said to be spontaneously broken if for some operator (which we will take to have spin 0 in order to preserve rotational symmetry) the vacuum expectation value is non-vanishing

$$\langle 0 | \Phi | 0 \rangle \equiv \Phi_0 \neq 0. \tag{2.10}$$

The operator Φ can be either a composite operator or a fundamental boson, and indeed we will see both kind of cases.

As now Φ has a non-vanishing value in the ground state, it makes sense to redefine the degrees of freedom in the field Φ into a modulus and a phase[2] as

$$\Phi = e^{i\varphi(x)}(\Phi_0 + \delta\Phi(x)). \tag{2.11}$$

Finiteness of energy requires $\varphi(x)$ to approach a constant value as $|\mathbf{x}| \to \infty$. Now states with different values of φ at infinity are equally good ground states of the system. Thus, there is a shifting symmetry $\varphi \to \varphi + a$, for constant a. Due to (2.10) this is not a symmetry of the ground state, but it is a symmetry of the low energy effective action for φ [11].

Thus, we see that spontaneous symmetry breaking leads to a scalar field φ with a shifting symmetry. We have not shown that the fluctuations of the modulus or possible other degrees of freedom satisfy the Landau criterion. This has to be checked case by case. If they do, then spontaneous symmetry breaking at finite charge density leads to superfluidity.

What are the symmetries that get spontaneously broken in real world superfluids? They are accidental global symmetries in the low energy physics. For example in Helium II, the conserved charge in question is the number of Helium atoms. This is a symmetry because the energy scales (and temperatures) of the experiment are low enough so that the helium atoms cannot disintegrate into other matter. Also in trapped atomic gases the conserved charge is the number of atoms, which is again conserved for the same reason.

[2] This is not sensible when expanding around $\Phi = 0$ as the phase of the complex number 0 is ill defined.

Chapter 3

Examples of Superfluids

In this section we review the simplest theories describing non-relativistic superfluids that are relevant to the real world. Eventually we will want to study a holographic model of superfluidity. That model is relativistic (at zero density and temperature) and has two spatial dimensions. So it is certainly different from the theories that we review in this section. The latter are the simplest and most well studied, so it is useful to understand them first.

3.1 Bosonic superfluids

We start from the simplest case, a single massive spinless boson Φ carrying a conserved $U(1)$ charge. This simple model of a superfluid is relevant for example in the description of Bose-Einstein condensates (BEC) of trapped bosonic atoms. Thus, we will refer to the model as the BEC theory.

We assume that there are bosons with a weak repulsive interaction at finite charge density[1]. We would like to set up an effective field theory to describe the system. The finite charge density in the effective field theory is accomplished by introducing a chemical potential μ coupled to the charge. The repulsive interaction can be modeled by a pointlike interaction $\lambda|\Phi|^4$, where $\lambda > 0$. This description is applicable as long as the energy scales and the density of bosons are low enough compared to the internal structure of the atoms [15, 16].

The full action for the boson is

$$S_{BEC} = \int dt d^3x \Big(\Phi^* (i\partial_t + \mu + \frac{\nabla^2}{2m}) \Phi - \lambda |\Phi|^4 \Big). \tag{3.1}$$

When studying scaling to low energies it is useful to use the non-relativistic scalings where one counts momentum having dimension 1 and energy having

[1] By charge density, we mean the global $U(1)$ charge corresponding to the conserved boson number.

dimension 2. In these units λ has dimension -1. This means that the system is getting weakly coupled at low energies and can be studied semiclassically. We can see this by forming a dimensionless coupling $\tilde{\lambda} = q\lambda$, where q is a typical momentum scale in the system, and seeing how the dimensionless coupling changes as we change the typical momentum scale

$$\beta_{\tilde{\lambda}} = q\frac{\partial\tilde{\lambda}}{\partial q} = \tilde{\lambda}. \tag{3.2}$$

This tells us that indeed the repulsive interactions ($\lambda > 0$) are getting weaker at low energies and momenta.

To find the ground state of the system we look for the minimum of the potential energy for a spacetime independent field Φ. The minimum is at

$$|\langle\Phi\rangle| = \sqrt{\frac{\mu}{2\lambda}}. \tag{3.3}$$

The ground state is no longer invariant under the $U(1)$ transformations generated by the charge

$$Q = \int d^3x|\Phi|^2, \tag{3.4}$$

and the symmetry is spontaneously broken. The ground state charge density is given by

$$\bar{n} = |\langle\Phi\rangle|^2 = \frac{\mu}{2\lambda}. \tag{3.5}$$

Next we would like to check that the excitation spectrum satisfies Landau's criterion. Rather than working with the action we will work directly with the equations of motion as is conventional when discussing the so called superfluid hydrodynamics. We parametrize the boson field in terms of polar coordinates in field space

$$\Phi = \sqrt{n}e^{i\phi}. \tag{3.6}$$

Here it is assumed that the density $n(x)$ is a function of space and time, and has a non-vanishing value in the ground state. The equation of motion following from (3.1) is

$$(i\partial_t + \mu + \frac{\nabla^2}{2m} + \lambda|\Phi|^2)\Phi = 0. \tag{3.7}$$

Multiplying the above equation with Φ^* and looking at the real and the imaginary part of the resulting equation one finds

$$\begin{aligned} \partial_t n + \nabla\cdot(n\mathbf{v}_s) &= 0, \\ m\partial_t\mathbf{v}_s + \nabla\Big(-\mu + \lambda n - \frac{1}{2m\sqrt{n}}\nabla^2\sqrt{n} + \frac{1}{2}m\mathbf{v}_s^2\Big) &= 0, \end{aligned} \tag{3.8}$$

where we have operated on the lower equation with ∇ and defined $\mathbf{v}_s = \nabla\phi/m$. The above equations look very similar to usual hydrodynamic equations of a fluid with a fluid velocity $\mathbf{v}_s$. Here $\mathbf{v}_s$ is an irrotational velocity field, which is called superfluid velocity. To find the excitation spectrum we look at small fluctuations around the vacuum $n = \bar{n} + \delta n$, $\mathbf{v}_s = \mathbf{0} + \delta\mathbf{v}$. Furthermore, by Fourier transforming we find that there is a mode with dispersion relation

$$\epsilon_p^2 = c_s^2\mathbf{p}^2 + \mathcal{O}(\mathbf{p}^4), \tag{3.9}$$

where the velocity of the Goldstone mode, which is often called the sound velocity, is

$$c_s = \sqrt{\frac{\mu}{m}}. \tag{3.10}$$

3.2 Bardeen-Cooper-Schrieffer (BCS) superfluids

In the last subsection we saw how bosons at finite density lead to superfluidity. In this section we will see that having fermions at finite density also leads fairly generically to superfluidity. In this case, superfluidity appears through dimensional transmutation. We will consider the following action

$$S_{BCS} = \int d^3x dt \Big(\Psi_\alpha^\dagger (i\partial_t + \frac{\nabla^2}{2m} + \mu)\Psi_\alpha - \lambda\Psi_+^\dagger\Psi_-^\dagger\Psi_-\Psi_+\Big). \tag{3.11}$$

Here Ψ_α is a fermion field and $\lambda|\Psi|^4$ is a local interaction, which is attractive for $\lambda < 0$. The index α denotes the spin degree of freedom. Naively, it seems that the coupling λ is irrelevant, since it again has dimension -1 and by a similar argument as in the preceding section we would conclude that the interaction is getting weak at low energies. This would mean that the system would behave as a free Fermi gas at low energies. This is not quite right though. In the case of fermions it is a bit more tricky to define what it means to scale to low energies [10]. Let us first consider the case $\lambda = 0$. Then the ground state is the Fermi sphere where all states with $|\mathbf{k}| < |\mathbf{k}_F|$ are occupied and the other states are unoccupied. Here we defined the Fermi momentum through $\mathbf{k}_F^2/2m = \mu$. The lowest energy fermionic excitations thus are either particles or holes with momenta close to the Fermi momentum.

Low energy thus means that the momenta are scaled towards the Fermi momentum. Next we would like to turn on the interaction. To proceed it is convenient to write the interaction term in momentum space as (note that time is not Fourier transformed)

$$\lambda \int dt \prod_{i=1}^{4} \frac{d^3\mathbf{k}_i}{(2\pi)^3}(2\pi)\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\Psi_+^\dagger(\mathbf{k}_4)\Psi_-^\dagger(\mathbf{k}_3)\Psi_-(\mathbf{k}_2)\Psi_+(\mathbf{k}_1). \tag{3.12}$$

Next we write $\mathbf{k}_i = \mathbf{q}_i + \mathbf{l}_i$, where $\mathbf{q}_i$ is the projection of $\mathbf{k}_i$ to the Fermi surface. To get to low energies we should do the following scalings [10]

$$\mathbf{l}_i \to s\mathbf{l}_i, \quad t \to s^{-1}t, \quad \Psi_\alpha \to s^{-1/2}\Psi_\alpha, \quad \mathbf{q}_i \to \mathbf{q}_i, \tag{3.13}$$

where $s < 1$ and eventually will be taken to zero. Here the scaling of the field Ψ was determined from the invariance of the kinetic term. This means that we are assuming that the kinetic term determines the size of generic fluctuations at low energies [10]. Also, near the Fermi surface, the kinetic energy of a particle becomes $\epsilon(\mathbf{k}) - \mu \approx l v_F$, where $v_F = \partial\epsilon/\partial k|_{\mathbf{k}=\mathbf{k}_F}$. Using the dispersion relation $\epsilon(\mathbf{k}) = \mathbf{k}^2/2m$ gives $v_F = k_F/m$. Using the above scalings (3.13) we see that the interaction term (3.12) for generic values of $\mathbf{k}_i$ scales as $s^{4-1-4/2} = s^1$. Here we assumed that the delta function does not contribute to the scaling as

$$\begin{aligned}&\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) = \delta(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4 + \mathbf{l}_1 + \mathbf{l}_2 - \mathbf{l}_3 - \mathbf{l}_4) \to \\ &\delta(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4 + s(\mathbf{l}_1 + \mathbf{l}_2 - \mathbf{l}_3 - \mathbf{l}_4)) \approx \delta(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4)\end{aligned} \tag{3.14}$$

as s is taken to be small. By this reasoning it would seem that as we go to low energies $s \to 0$, the interaction term scales as s, so that it will become irrelevant. This is still not quite right since the scaling (3.14) is not generally right. Consider the case when $\mathbf{k}_1 + \mathbf{k}_2 = 0$. Then the delta function becomes

$$\delta(-\mathbf{k}_3 - \mathbf{k}_4) = \delta(-\mathbf{q}_3 - \mathbf{q}_4 - \mathbf{l}_3 - \mathbf{l}_4). \tag{3.15}$$

But now the parts of the delta function perpendicular to $\mathbf{q}_3$ will constrain $\mathbf{q}_3 + \mathbf{q}_4 = 0$, there is a one dimensional delta function left of the form $\delta(-l_3 - l_4)$ which scales as

$$\delta(-l_3 - l_4) \to \delta(-s(l_3 + l_4)) = s^{-1}\delta(-l_3 - l_4). \tag{3.16}$$

So overall, the interaction term scales as s^0 when the initial states are in opposite sides of the Fermi sphere and thus, the interaction is marginal.

To determine whether the interaction is marginally irrelevant or relevant one has to compute loop corrections to the 4-point amplitude. A computation of the one loop correction [10] shows that there is a non-vanishing beta function

$$\beta_\lambda = q\frac{d\lambda}{dq} = N\lambda^2, \tag{3.17}$$

where N is the density of states at the Fermi surface

$$N = \int \frac{d^2\mathbf{q}}{(2\pi)^3}\frac{1}{v_F}. \tag{3.18}$$

Integrating this, gives

$$\lambda(q) = \frac{\lambda(q')}{1 + N\lambda(q')\log(q'/q)}. \tag{3.19}$$

We see that the repulsive interactions $\lambda > 0$ are getting weak at low energies, while attractive interactions $\lambda < 0$ get strong. The momentum scale Λ_s at which the interaction becomes strong can be estimated by setting $\lambda(\Lambda_s) = \infty$ and solving for Λ_s in (3.19) to give the dynamically generated scale

$$\Lambda_s \propto q e^{1/(N\lambda(q))}. \tag{3.20}$$

Generally as a quantum field theory gets strongly coupled things get difficult to calculate. In the case of BCS theory this is not the case. If we assume that the fermions form pairs (called Cooper pairs) which condense, which is a phenomenon that seems fairly generic in nature, things become nicely calculable. The reason for the simplicity is due to the kinematics of the Cooper pairs. The fact that the fermion interactions are strong only when the fermions sit at opposite sides of the Fermi sphere, allows one to neglect loops of Cooper pairs since they cannot carry non-zero total momentum.

Next we can go to real calculations [17]. Consider doing a path integral over the fermions

$$Z = \int [d\Psi_\alpha, d\Psi_\alpha^\dagger] e^{iS_{BCS}}. \tag{3.21}$$

The problem with performing the integral is the quartic interaction. There is a nice trick called the Hubbard-Stratanovich transformation [18, 19], that one can use to get rid of the non-linearity. One can introduce a new field Δ and use the identity

$$\begin{aligned} & e^{-i\int d^3x dt \lambda \Psi_+^\dagger \Psi_-^\dagger \Psi_- \Psi_+} \\ &= \int [d\Delta, d\Delta^*] \exp\Big[-i\int d^3x dt \Big(\Delta^*\Psi_-\Psi_+ + \Delta\Psi_+^\dagger\Psi_-^\dagger - \frac{1}{\lambda}|\Delta|^2\Big)\Big]. \end{aligned} \tag{3.22}$$

This allows us to write

$$\begin{aligned} Z = & \int [d\Psi_\alpha, d\Psi_\alpha^\dagger, d\Delta, d\Delta^*] \exp\Big[i\int d^3x dt \Big(\Psi_\alpha^\dagger (i\partial_t + \frac{\nabla^2}{2m} + \mu)\Psi_\alpha \\ & - \Delta^*\Psi_-\Psi_+ - \Delta\Psi_+^\dagger\Psi_-^\dagger + \frac{1}{\lambda}|\Delta|^2\Big)\Big] \end{aligned} \tag{3.23}$$

$$= \int [d\Delta, d\Delta^*] e^{iS_{eff}}. \tag{3.24}$$

On the second line we have integrated out the fermion fields, which is now easy since the action was quadratic in the fermion fields. To see what Δ means we can use the fact that the functional integral of a total derivative is zero

$$0 = \int [d\Psi_\alpha, d\Psi_\alpha^\dagger, d\Delta, d\Delta^*] \frac{\delta}{\delta\Delta^*} e^{iS[\Delta,\Psi]}, \tag{3.25}$$

where S is the action appearing in (3.23). Taking the functional derivatives leaves us with the relation

$$\langle(\Delta(x) - \lambda\Psi_-(x)\Psi_+(x))\rangle = 0. \tag{3.26}$$

So the vacuum expectation value of Δ is identical to the vacuum expectation value of the fermion bilinear $\Psi_-\Psi_+$. Also we see that Δ carries twice the charge of the fermion under the global $U(1)$ symmetry corresponding to conserved fermion number. The effective action for Δ in (3.24) has the form

$$S_{eff} = \mathrm{Tr}\log K + \frac{1}{\lambda}\int d^3x dt|\Delta|^2, \tag{3.27}$$

where K is the fermion kernel which we will come back to in a moment. As we discussed, the fermion interactions are marginally relevant only when the total momentum of a fermion pair $\Psi_-\Psi_+$ is zero. This means that we are allowed to neglect doing loops in the Δ path integral, since they would involve fermion pairs carrying non-zero momenta. This approximation is often called the mean field approximation. This way the vacuum expectation value for Δ is the classical saddle point of the effective action (3.27) given by

$$\frac{\delta}{\delta\Delta^*}\mathrm{Tr}\log K = -\frac{1}{\lambda}\Delta. \tag{3.28}$$

One can easily work out the operator K in the Nambu basis $\Psi = (\Psi_+, \Psi_-^*)$,

$$K = \begin{pmatrix} i\partial_t - \epsilon(-i\nabla) & -\Delta \\ -\Delta^* & i\partial_t + \epsilon(-i\nabla) \end{pmatrix}. \tag{3.29}$$

In this way, (3.28) becomes

$$\frac{1}{\lambda} = -i\int \frac{dp_o d^3\mathbf{k}}{(2\pi)^4}\frac{1}{p_0^2 - \epsilon^2(\mathbf{k}) - |\Delta|^2 + i\varepsilon}. \tag{3.30}$$

Here we introduced an imaginary part $i\varepsilon$ to pick the vacuum state and to make the path integral to converge [20, 21]. The p_0 integration picks up a single residue so we get

$$\frac{1}{\lambda} = -\frac{1}{2}\int \frac{d^2\mathbf{q}}{(2\pi)^3}\int_0^\Lambda dl \frac{1}{\sqrt{v_F^2 l^2 + |\Delta|^2}}. \tag{3.31}$$

This is called the gap equation. The l integral is an elementary integral that can be performed to give

$$\frac{1}{\lambda} = -\frac{1}{2}N\mathrm{arcsinh}\Big(\frac{\Lambda v_F}{|\Delta|}\Big), \tag{3.32}$$

where N is again the density of states on the Fermi surface and we have used the fact that our Fermi surface is spherical so that $v_F = k_F/m$ is independent of $\mathbf{q}$. Assuming for the moment that the cutoff Λ may be taken a to be a lot larger than $|\Delta|/v_F$ we can approximate

$$\frac{1}{\lambda} \approx -N\log\Big(\frac{\Lambda v_F}{|\Delta|}\Big), \tag{3.33}$$

and we can solve for the vacuum expectation value

$$|\Delta| \approx \frac{k_F \Lambda}{m} e^{1/\lambda N}. \tag{3.34}$$

Here we see that the condensate is indeed proportional to the dynamically generated scale obtained from the renormalization group considerations (3.20). Also we see that the assumption $\Lambda \gg |\Delta|/v_F$ is well justified as long as the UV coupling λ is sufficiently small.

Indeed we see that the Cooper pairs condense and spontaneously break the $U(1)$ symmetry that rotates $\Delta \to e^{2i\alpha}\Delta$. This inevitably leads to Goldstone bosons from the phase fluctuations of Δ.

A simple way to see what happens to the fermionic excitations when the condensate forms is to go back to (3.23) and substitute the vacuum expectation value for Δ into the action. Clearly this will make the fermions gapped. Classically the fermions satisfy

$$K\Psi = 0, \tag{3.35}$$

which upon Fourier transformation leads to the Fermion excitation spectrum

$$E(\mathbf{k}) = \pm\sqrt{\epsilon^2(\mathbf{k}) + |\Delta|^2}. \tag{3.36}$$

So indeed the fermion excitations have a gap $|\Delta|$.

To obtain the Goldstone spectrum we can expand the fermions as $\Psi_\alpha = e^{\imath\phi}\chi_\alpha$. This changes the operator K into

$$K = \begin{pmatrix} i\partial_t - \partial_t\phi - \epsilon(-i\nabla + \nabla\phi) & -\Delta \\ -\Delta^* & i\partial_t + \partial_t\phi + \epsilon(-i\nabla - \nabla\phi) \end{pmatrix}. \tag{3.37}$$

Expanding (3.27) (with K given above) in powers of derivatives acting on ϕ one obtains an effective action for ϕ. From that effective action one can read off the sound velocity [17]

$$c_s = \frac{1}{\sqrt{3}} v_F. \tag{3.38}$$

3.3 A crossover between BCS and BEC and scale invariant superfluids

In the preceding section we concluded that fermions with weak attractive interactions at finite density lead to a BCS superfluid. Since the fermion interactions for fermions not at opposite points of the Fermi sphere were irrelevant it seems like the BCS kind of superfluid is the only possibility for the long distance physics.

Up to this point we have been following the renormalization group to low energies. Now we will give up renormalization group and discuss what happens when one tunes the fermion interactions. This is also possible experimentally since one can control the interactions of ultracold fermions using the Feshbach resonance [23].

If one starts increasing the fermion interactions in a BCS superfluid, the fermions will form bound states and eventually the bound state bosons condense due to Bose-Einstein condensation. This way one can go from a BCS superfluid to a BEC superfluid. An interesting question is what happens in between the two superfluids. One finds that there is a continuous transition between the two types of superfluids [23]. The most interesting physics is found right at the point where a two fermion bound state appears at zero energy. The treshold boundstate makes the renormalized fermion interaction to diverge $\lambda_R \to \infty$, meaning that the scattering length grows larger than any other scales in the problem. The only scales in the problem are the fermion density and possibly the temperature (if it is non-vanishing). This system is believed to be described by a non-relativistic conformal field theory [24]. The fixed point describing the system is not infrared stable [25] so we would not have seen it by considering the low energy limit of BCS theory.

As we will see in the following sections, the holographic duality is a promising approach to study strongly coupled and scale invariant systems. For this reason one might hope to learn more about scale invariant superfluids in the BCS-BEC crossover by using holographic methods.

3.4 The two-fluid model

So far we have been treating superfluids as fluids with vanishing viscosity. However, real world superfluids usually consist of two components, a superfluid with a vanishing viscosity and a normal fluid with a non-vanishing viscosity. This view of a superfluid is called the two fluid model [7, 8]. In this section we review some very basics of the two fluid model in the context of the BEC theory, following [17].

First consider a superfluid moving with a velocity $\mathbf{u}$ with respect to the laboratory frame. Next we perform a Galilean boost to the rest frame of the superfluid. The overall effect of this is to replace $\partial_t \to \partial_t + \mathbf{u} \cdot \nabla$. Plugging this to the BEC action gives

$$S_{BEC} = \int d^3x dt \Big(\Phi^* (i\partial_t - \mathbf{u} \cdot (-i\nabla) + \mu + \frac{\nabla^2}{2m}) \Phi - \lambda |\Phi|^4 \Big). \tag{3.39}$$

In section (3.1) we saw that the velocity of the superfluid is given by the gradient of the phase of the condensate wavefunction. To have a flow where the superfluid velocity is different from $\mathbf{u}$ we replace $\Phi \to e^{im\mathbf{v}_s \cdot \mathbf{x}} \Phi$. Plugging this into (3.39) gives

$$S_{BEC}(\mathbf{u},\mathbf{v}_s) = \int d^3\mathbf{x}dt\Big(\Phi^*(i\partial_t - \mathbf{u}\cdot(-i\nabla) + \mu_{eff} + \frac{\nabla^2}{2m}$$
$$+ (\mathbf{u}-\mathbf{v}_s)\cdot(-i\nabla))\Phi - \lambda|\Phi|^4\Big), \tag{3.40}$$

where

$$\mu_{eff} = \mu - \frac{1}{2}m\mathbf{v}_s\cdot(\mathbf{v}_s - 2\mathbf{u}). \tag{3.41}$$

Next we would like to calculate the finite temperature effective potential

$$e^{-V_3\beta V_{eff}} = \int [d\phi]e^{-S_{BEC}^{(Euc)}(\mathbf{u},\mathbf{v}_s)}. \tag{3.42}$$

The one loop evaluation of the above path integral gives

$$V_{eff} = -\frac{\mu_{eff}}{2\lambda} + \frac{1}{2V_3\beta}\mathrm{Tr}\log K', \tag{3.43}$$

where

$$K' = \begin{pmatrix} \partial_\tau - \epsilon(i\nabla) + \mu_{eff} - 4\lambda|\phi_0|^2 & -2\lambda\phi_0^2 \\ -2\lambda(\phi_0^*)^2 & -\partial_\tau - \tilde{\epsilon}(i\nabla) + \mu_{eff} - 4\lambda|\phi_0|^2 \end{pmatrix}. \tag{3.44}$$

Above we have defined $\epsilon(i\nabla) = -(\mathbf{u}-\mathbf{v}_s)\cdot(-i\nabla) - \frac{\nabla^2}{2m}$ and $\tilde{\epsilon}(i\nabla) = (\mathbf{u}-\mathbf{v}_s)\cdot(-i\nabla) - \frac{\nabla^2}{2m}$. The determinant of K' can be calculated using standard methods of finite temperature field theory [26]

$$V_{eff} = -\frac{\mu_{eff}}{2\lambda} + \frac{1}{2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\Big(E(\mathbf{k}) + 2T\log(1 - e^{-(E(\mathbf{k})+(\mathbf{u}-\mathbf{v}_s)\cdot\mathbf{k})/T})\Big), \tag{3.45}$$

where

$$E(\mathbf{k}) = \sqrt{(\frac{\mathbf{k}^2}{2m} + 4\lambda|\phi_0|^2 - \mu_{eff})^2 - 4\lambda^2|\phi_0|^4}. \tag{3.46}$$

We can calculate the momentum density of the flow by taking a derivative of the effective potential with respect to $\mathbf{u}$. This follows from the fact that $\mathbf{u}$ multiplies the total momentum operator in (3.39). Denoting the momentum density as $\mathbf{p}$ we get

$$\mathbf{p} = \frac{\partial V_{eff}}{\partial \mathbf{u}} = \frac{\partial \mu_{eff}}{\partial \mathbf{u}}\frac{\partial V_{eff}}{\partial \mu_{eff}} - \int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{\mathbf{k}}{e^{(E(\mathbf{k})+(\mathbf{u}-\mathbf{v}_s)\cdot\mathbf{k})/T} - 1}$$
$$= m\bar{n}\mathbf{v}_s - \int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{\mathbf{k}}{e^{(E(\mathbf{k})+(\mathbf{u}-\mathbf{v}_s)\cdot\mathbf{k})/T} - 1}. \tag{3.47}$$

Here we used the identity $\bar{n} = \partial V_{eff}/\partial \mu_{eff}$. Let us first consider the limit $T \to 0$. In this limit the second term, which is due to finite temperature excitations, vanishes. This way we get

$$\mathbf{p} = m\bar{n}\mathbf{v}_s. \tag{3.48}$$

So at zero temperature the superfluid part is the only thing flowing. Assuming that $|\mathbf{u} - \mathbf{v}_s| \ll 1$, we can expand (3.47) in the velocity difference. To the first order, we get

$$\mathbf{p} = \rho\mathbf{v}_s + \rho_n(\mathbf{u} - \mathbf{v}_s), \tag{3.49}$$

where we have defined the density of the finite temperature excitations as

$$\rho_n = \frac{1}{3T}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{\mathbf{k}^2 e^{E(\mathbf{k})/T}}{(e^{E(\mathbf{k})/T} - 1)^2}. \tag{3.50}$$

We will call ρ_n as the "normal" density. Identifying $\mathbf{u} = \mathbf{v}_n$ as the velocity of the normal part of the fluid and defining a superfluid density as

$$\rho_s = \rho - \rho_n, \tag{3.51}$$

we get the momentum density into the form

$$\mathbf{p} = \rho_s\mathbf{v}_s + \rho_n\mathbf{v}_n. \tag{3.52}$$

This way we are led to a version of the two fluid model pioneered by Landau where there is a superfluid and a normal fluid having their own densities and independent velocities. The normal fluid part satisfies the hydrodynamic equations of a normal viscous fluid, while the superfluid behaves like an ideal irrotational fluid, as we saw in (3.8). Now after getting some taste of the simplest models of superfluids we are ready to move on to the actual topic of the thesis.

Chapter 4

Holography

4.1 Degrees of freedom in a quantum field theory

For any quantum field theory which is supposed to make sense at short distances, there has to be a UV fixed point in the renormalization group, at least according to the usual Wilsonian picture of quantum field theory [27]. This means that the theory becomes scale invariant at high energies. As far as the high energy states of such a system are concerned, it behaves as a conformal field theory (CFT).

For a local quantum field theory, the Hamiltonian is an extensive operator. This, together with dimensional analysis, implies that in a finite temperature CFT the energy expectation value $\langle H \rangle = E$ scales with the temperature as

$$E \propto VT^d, \tag{4.1}$$

where d is the number of spacetime dimensions and V is the spatial volume of the system. For the conventional density matrix $e^{-\beta H}$, the extensivity of the Hamiltonian implies the extensivity of entropy. Thus, dimensional analysis tells us that the entropy of a CFT behaves as

$$S \propto VT^{d-1}. \tag{4.2}$$

This way we see that the entropy of a finite temperature CFT behaves as a function of energy as

$$S \propto E^{(d-1)/d}. \tag{4.3}$$

So for any UV complete quantum field theory the entropy at large energies behaves as (4.3).

4.2 Degrees of freedom in gravity

Next we would like to determine the high energy entropy of a theory of gravity. In asymptotically flat space, the finite temperature collapses the whole system into a single infinitely large black hole [28]. To make more sense of the canonical ensemble we can work at a spacetime which is not asymptotically flat, but has negative constant curvature at "infinity". Such a spacetime is called Asymptotically-Anti-de-Sitter ($AAdS$). What saves $AAdS$ spacetimes from collapsing to an infinitely large black hole at finite temperature is that $AAdS$ acts as a gravitational confining potential to the matter.

The metric of the AdS spacetime is

$$ds^2 = \frac{dr^2}{1+\frac{r^2}{L^2}} - \Big(1+\frac{r^2}{L^2}\Big)dt^2 + r^2 d\Omega^2, \tag{4.4}$$

where L is the curvature radius of the spacetime and $d\Omega^2$ is the metric on a unit $(d-2)$-sphere. The metric of an $AAdS$ spacetime will approach the form (4.4) as $r \to \infty$. Due to the warp factor in front of dt^2, the local temperature in $AAdS$ behaves as $T_{loc} = T/\sqrt{g_{tt}} \propto T/r$ for large r and finite temperature can be achieved with finite energy [28]. As one puts more and more energy into AdS space (or increases temperature) eventually one will form black holes. The entropy of a single black hole is given by the Bekenstein-Hawking formula (we will come back to the derivation of this result in the next section)

$$S_{BH} = \frac{A}{4G_N}, \tag{4.5}$$

where A is the area of the black hole horizon and G_N is the Newton's constant. At temperatures larger than the critical temperature of the Hawking-Page phase transition [28] the thermodynamically favored state is a single large black hole with a metric [27]

$$ds^2 = \frac{dr^2}{f(r)} - f(r)dt^2 + r^2 d\Omega^2, \quad f(r) = 1 - \frac{16\pi G_N M}{(d-2)\mathrm{Vol}(S^{d-2})r^{d-3}} + \frac{r^2}{L^2}, \tag{4.6}$$

where $\mathrm{Vol}(S^{d-2})$ denotes the volume of a $d-2$ sphere with a unit radius. For high energies (or equivalently large M), the position of the black hole horizon is of the form

$$r_0 \propto (ML^2G_N)^{1/(d-1)}. \tag{4.7}$$

Identifying the black hole mass with the total energy of the system, we obtain the black hole entropy

$$S_{BH} \propto r_0^{d-2} \propto M^{(d-2)/(d-1)} \propto E^{(d-2)/(d-1)}. \tag{4.8}$$

By comparing (4.3) and (4.8), we see that the number of high energy states of a gravitational theory seem to behave very differently from the behavior of a local quantum field theory. This is one argument to suggest that gravity cannot be a renormalizable quantum field theory [27] since it does not have the correct number of degrees of freedom to be a CFT in the UV.

By comparing (4.3) and (4.8), we see that the number of high energy degrees of freedom in a gravitational theory behave as those of a CFT in spacetime with one less dimension. This leads to the idea of holography, that gravitational theories could be dual to local quantum field theories with one less spacetime dimension [29, 30]. The first precise version of such a duality was conjectured in the framework of string theory by Maldacena [1]. Since this is the best understood case of the duality, we will review it to obtain some generic lessons.

4.3 Maldacena's duality

In this section we review the duality between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Type IIB string theory has as its low energy limit, the type IIB supergravity. Type IIB supergravity has the following bosonic fields, the graviton $g_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$, a scalar called dilaton Φ, and 3 independent Ramond-Ramond p-form fields C_0, C_2, C_4.

Type IIB supergravity has well known classical solutions that are charged under the Ramond-Ramond fields. The relevant classical solution to us is the D3-brane solution [31]

$$\begin{aligned} ds^2 &= \sqrt{H(r)}(dr^2 + r^2 d\omega_5^2) + \frac{1}{\sqrt{H(r)}}(-dt^2 + d\mathbf{x}^2), e^{\Phi} = 1, \\ H(r) &= 1 + \frac{4\pi g_s N l_s^4}{r^4}, \end{aligned} \tag{4.9}$$

and the C_4 field satisfies $\int dC_4 = N$. Other fields vanish for this solution. This looks like a usual extremal black brane solution in supergravity. The amazing thing in string theory is that in addition to being a classical black hole solution, the D3-brane has an interpretation directly in string perturbation theory as an object where open strings can end [32].

First we will consider the perturbative picture of D-branes and take the low energy limit $\alpha' E^2 \to 0$, where E is a typical energy scale in the problem. The dynamics of a single D3-brane is given by a generalization of the Dirac-Born-Infeld (DBI) action of a membrane. The bosonic part of the D3-brane action is [33]

$$S_{D3} = -\frac{1}{g_s (2\pi)^3 (\alpha')^2} \int d^4\sigma e^{-\Phi} \sqrt{\det |g_{ab} + B_{ab} + 2\pi\alpha' F_{ab}|} + S_{CS}, \tag{4.10}$$

where g_{ab} and B_{ab} are the pullbacks of the corresponding spacetime fields to the D3-brane worldvolume. F_{ab} is the field strength of a $U(1)$ gauge field living on the D3-brane. S_{CS} is a Chern-Simons like term involving the worldvolume gauge field and the p-form gauge fields whose explicit form we will not specify here, but refer to [33]. The embedding of the D3-brane into spacetime can be parametrized with coordinates $X^\mu(\sigma^a)$, where σ^a are coordinates on the brane worldvolume. In the above action we can choose a gauge, called the static gauge, where $\sigma^a = X^a \equiv x^a$, for $a = 0, ..., 3$ and $X^M = X^M(x^a)$ for $M = 4, 5, ..., 9$. Thus $X^M(x^a)$ denotes the position of the D3-brane in the 6 transverse directions. When the fields X^M and A_μ are slowly varying we can expand the action in powers of derivatives. Furthermore assuming a flat backround with vanishing B_{ab} and Φ fields, we get

$$S_{D3} = -\frac{1}{g_s(2\pi)^3(\alpha')^2}\int d^4x\Big(1 + \frac{1}{2}\partial_\mu X^M\partial^\mu X^M + \frac{(2\pi\alpha')^2}{4}F_{\mu\nu}F^{\mu\nu} + ... - 1\Big), \tag{4.11}$$

where the dots denote terms with more than two derivatives. These terms are to be dropped in the low energy limit. The last term -1 factor comes from a Chern-Simons term of the form

$$\mu_3\int C_4, \tag{4.12}$$

where μ_3 is the RR charge of the D3-brane, which due to a supersymmetric BPS condition[1] exactly cancels the first term which corresponds to the mass of the D3-brane. Such a cancellation may be seen to follow from supersymmetry since supersymmetry requires the vacuum energy to vanish. The dynamics of the fermions is fixed by supersymmetry. The number of supersymmetries in type IIB string theory is 32, which is broken by the presence of the D3 brane into 16 supercharges. This is because the presence of D3-branes requires the existence of open string states for which the total of 32 supercharges is reduced to half because the left and right moving supersymmetries become dependent for the open strings. 16 supercharges corresponds to $\mathcal{N} = 4$ supersymmetry on the 3+1 dimensional brane worldvolume. Thus, we can identify the low energy dynamics of a single D3-brane as $\mathcal{N} = 4$ supersymmetric QED with a gauge coupling

$$g^2_{QED} = 2\pi g_s. \tag{4.13}$$

A single D3-brane carries a single unit of the 5-form charge. So the solution (4.9) really describes N D3-branes on top of each other. The low energy dynamics of a stack of N D3-branes is given by a non-Abelian version of (4.10), which can be similarly expanded in powers of derivatives [35]. This time the low energy dynamics is given by $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with a gauge group $U(N)$ with the gauge coupling

$$g^2_{YM} = 2\pi g_s. \tag{4.14}$$

[1] By a BPS condition we mean here that there is a specific linear relation between a gauge charge and the mass of a state which follows from supersymmetry [34].

The bosonic part of the action of this theory is [33]

$$S_{bos} = \frac{1}{g_{YM}^2} \int d^4x \mathrm{Tr}\Big(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu \Phi^I D^\mu \Phi^I - \frac{1}{4} [\Phi^I, \Phi^J]^2 \Big), \quad (4.15)$$

where we have rescaled the scalar fields $\Phi^I = (2\pi\alpha')^{-1} X^I$. The important points to us here are that there is a $U(N)$ gauge symmetry with A_μ and Φ in the adjoint representation, and there is a global symmetry that rotates $\Phi^I \to R^{IJ}\Phi^J$, where R^{IJ} is an $SO(6)$ matrix. Taking into account the fermionic part of the action one sees that the fermions also transform under this symmetry since there is a Yukawa coupling between the scalars and the fermions. This global $SO(6)$ symmetry is called an R-symmetry since it is equivalent to rotating the 4 supersymmetry generators of $\mathcal{N} = 4$ with an $SU(4)$ transformation (recalling that $SU(4) \cong SO(6)$). This symmetry follows simply from the rotational $SO(6)$ symmetry in the directions orthogonal to the D3-branes.

We have obtained a low energy description for a stack of D3-branes in terms of a quantum field theory living on its worldvolume. Interestingly there is a second way to take the low energy limit $E^2\alpha' \to 0$ [1]. Alternatively we can start from the classical supergravity solution (4.9). Far away from the brane the spacetime looks flat and the $\alpha' \to 0$ leaves us with free supergravity in flat space. More interesting things happen when we approach $r = 0$. Close to $r = 0$ the metric in (4.9) becomes

$$ds^2 = L^2\Big(\frac{dr^2}{r^2} + d\Omega_5^2\Big) + \frac{r^2}{L^2}(-dt^2 + d\mathbf{x}^2), \quad (4.16)$$

where

$$L^2 = \sqrt{4\pi g_s N}\alpha' = \sqrt{2g_{YM}^2 N}\alpha'. \quad (4.17)$$

We can recognize the metric (4.16) as $AdS_5 \times S^5$ where both factors have a curvature radius of magnitude L. The AdS part of the metric (4.16) is a different parametrization of anti-de Sitter space than that in (4.4). The metric (4.16) covers the Poincare patch of the AdS^5 spacetime. We are interested in the physics of modes with energies E, that satisfy $E^2\alpha' \to 0$. In the backround (4.16) the proper energy of such a mode is

$$E_{proper} = \frac{(2g_{YM}^2 N)^{1/4}\sqrt{\alpha'}}{r} E. \quad (4.18)$$

Thus, we see that the modes with $E^2\alpha' \to 0$ can have a non-zero proper energy if we at the same time take $r \to 0$. So we see that indeed there is non-trivial physics left in this limit. Next we can see whether the modes near the $r = 0$ region are interacting or not. Since the curvature radius of the region near

$r = 0$ is of the order of L the quantum gravitational corrections are controlled by the dimensionless Newton's constant

$$\frac{G_N^{(11)}}{L^8} \propto \frac{g_s^2(\alpha')^4}{(g_s N)^2(\alpha')^4} = \frac{1}{N^2}. \tag{4.19}$$

Thus, we see that as $\alpha' \to 0$ we are left with a gravitational theory with a finite gravitational coupling determined by N. Another quantity of interest is the dimensionless string scale, which controls the string loop corrections and is determined from the dimensionless ratio of α' and L as

$$\frac{l_s^2}{L^2} = \frac{\alpha'}{L^2} \propto \frac{1}{g_{YM}^2 N}. \tag{4.20}$$

Also we see that the dimensionless string scale is finite as $\alpha' \to 0$. We have been assuming in our derivation that both, the stringy effects and the string loop effects are small. Particularly this means that $g_s < 1$. In the perturbative description of D3-branes we also need $\lambda = g_{YM}^2 N < 1$. On the other hand, weak coupling in the supergravity description of the D3-branes meant that $\lambda \gg 1$ and $N \gg 1$.

To summarize we have seen that the physics of D3 branes can be described in two alternative ways in the low energy limit. It is fairly natural to assume that these two descriptions of the system are equivalent to each other [1]. Maldacena suggested that these two descriptions might be equivalent to each other for all values of λ and N.

To see how to make the duality more precise we take a few steps back and look at what happens before we strictly take $\alpha' \to 0$. Then it is more convenient to use a dimensionless AdS coordinate $u = r/\sqrt{\alpha'}$. The boundary of AdS space would be at $u \to \infty$. Now when $\alpha' \neq 0$, the boundary gets replaced by a transition to the flat space outside the D3-brane. If we send a fluctuation of a supergravity field towards the D3-brane, there are two ways to interpret what happens.

The first way is to use the perturbative picture of D-branes, in which case the fluctuation couples to the quantum field theory on the D3-branes and sources the quantum fields on the brane. So changing values of supergravity fields outside the D3-brane can be interpreted as varying sources on the $\mathcal{N} = 4$ SYM. The response of $\mathcal{N} = 4$ SYM to supergravity fields can be formalized by defining the generating functional

$$Z_{QFT}[J_i] = \int [dA_\mu, ...] e^{iS_{SYM} + i\int J_i \mathcal{O}_i}, \tag{4.21}$$

where $\mathcal{O}_i$ are operators in SYM to which the supergravity fields couple to and J_i are the values of the supergravity fields near the D3-branes.

Another way of interpreting what happens when we send fluctuations of the supergravity fields towards the branes is to use the supergravity solution. When $\alpha' \neq 0$ the fluctuation propagates into the $AdS_5 \times S^5$ region and excites the fields inside that region. Now as we take $\alpha' \to 0$, the large u region, corresponding to the region of space outside the branes, goes to $u \to \infty$. So we might identify the values of the supergravity fields ϕ_i at $u = \infty$ as the supergravity fields near the D3-branes as seen from the asymptotically flat region. This way we would make the identification $\phi_i(u = \infty) \propto J_i$.

We would like to identify an observable in supergravity (or string theory) with the SYM generating functional (4.21). For this identification to make sense, the gravitational observable should be a functional of J_i or equivalently of $\phi_i(u = \infty)$. The simplest gravitational observable would be the gravitational partition function with fixed boundary conditions at $u = \infty$

$$Z_{string}[J_i] = \int [d\phi_i] e^{iS_{string}}|_{\phi_i|_{\partial AdS} \propto J_i}. \tag{4.22}$$

This seems like a good observable in full quantum gravity as it is gauge invariant[2].

The key equation in AdS/CFT duality is the identification of the two generating functionals [37, 38]

$$Z_{QFT}[J_i] = Z_{string}[J_i]. \tag{4.23}$$

This is the basic relation we will work with.

As an example of the "dictionary" between bulk fields and SYM operators we can take the metric $g_{\mu\nu}$. If one expands the D3-brane action with a non-trivial backround metric, one sees that the backround metric couples to the energy momentum tensor of SYM. This also follows from symmetry, since a spin 2-particle should couple to a conserved two component tensor current. The energy momentum tensor is the only possible conserved two component tensor current. Thus we learn that the boundary value of the metric can be identified as the source for the energy momentum tensor in SYM. Correlation functions of the SYM energy momentum tensor can be obtained by taking functional derivatives of the string partition function with respect to the boundary value of the metric.

String theory in AdS is complicated. A much more simplified situation is obtained in the limit where $N \to \infty$ and $\lambda \to \infty$. Then the string loops and the stringy α' corrections become arbitrarily small so that one can hope to approximate the string path integral with its saddle point value in the supergravity approximation

$$Z_{string}[\phi_i] \approx e^{iS_{sugra}}, \tag{4.24}$$

[2]The gauge invariance in question here is the diffeomorphism invariance.

where the supergravity action is evaluated on the solutions of the corresponding equations of motion satisfying the right boundary conditions at $u \to \infty$. In this way one can obtain correlation functions in strongly coupled SYM by solving classical equations of motion in AdS space. In this approximation equation (4.23) reduces to

$$Z_{QFT} = \langle e^{i\int d^dx J^i \mathcal{O}_i} \rangle \approx e^{iS_{sugra}}. \tag{4.25}$$

The $N \to \infty$ limit corresponds to a classical limit in the gravitational theory. So the quantum fluctuations are suppressed by powers of N^{-1}. Somehow the quantum fluctuations in the SYM should be suppressed for the duality to make sense. This is related to the properties of the operators $\mathcal{O}_i$ in (4.21), in the large N limit. The operators $\mathcal{O}_i$ must be gauge invariant and involve a trace over the colour indices. For example one might have $\mathcal{O}_i = \mathrm{Tr}(\Phi^2)/N$. Such operators are called single trace operators, and they have an important factorization property [22] when properly normalized

$$\langle \mathcal{O}_i \mathcal{O}_j \rangle = \langle \mathcal{O}_i \rangle \langle \mathcal{O}_j \rangle + O(1/N^2). \tag{4.26}$$

So indeed the fluctuations in the expectation values of the single trace operators satisfy

$$\lim_{N\to\infty} \langle \mathcal{O}^2 - \langle \mathcal{O} \rangle^2 \rangle = 0. \tag{4.27}$$

Thus, the large N limit seems like a classical limit [39]. Still, this is not a classical limit in the usual meaning of the word. This can be seen in perturbation theory by noting that there is an infinite number of Feynman diagrams in SYM that contribute in the leading order in the $1/N$ expansion as $N \to \infty$ [40].

4.4 Realization of symmetries in the duality

Let us start from symmetries of the $AdS_5 \times S^5$ spacetime. There is an obvious $SO(6)$ symmetry rotating the S^5 which comes from the rotational symmetry on the directions orthogonal to the brane. This $SO(6)$ symmetry is seen as a global symmetry on SYM while it is a part of the local diffeomorphism symmetry in the bulk[3]. The metric of AdS_5 (4.16) has an $SO(3,2)$ symmetry. We can see some parts of it very easily. The $-dt^2 + d\mathbf{x}^2$ part has an obvious Lorentz symmetry $SO(3,1)$. Then there is a scaling symmetry that acts as

$$(t, \mathbf{x}, r^{-1}) \to \lambda(t, \mathbf{x}, r^{-1}), \tag{4.28}$$

where λ is an arbitrary non-zero real number. As one combines the Lorentz transformations and the scalings with a transformation that acts on the boundary as a special conformal transformation [31], they together form the group

[3] We often refer to the dual gravitational description as the bulk description and the field theory as the boundary theory.

$SO(3,2)$. This is equivalent to the conformal group in $3+1$ dimensions. This would better be a symmetry of the SYM for the duality to make sense. The $SO(3,1)$ part is simply the Lorentz symmetry of SYM. At the classical level SYM in fact enjoys the full conformal symmetry. Field theories with classical scale invariance may loose the symmetry because of quantum effects. In $\mathcal{N}=4$ SYM the beta function (which we recall as the measure of scale dependence of a coupling constant) is known to vanish in all orders of perturbation theory and probably also non-perturbatively [41]. Thus, the $SO(3,2)$ symmetry is also present at quantum level in the SYM.

At this point we mention an important point regarding to the mapping of parameters between the two sides of the duality. As one compactifies type IIB supergravity on the S^5 one obtains particles with non-vanishing masses in the bulk. Since the boundary theory should be scale invariant, there can be no mass scales involved in the theory. So what does the value of a mass of a particle in the bulk mean in the boundary field theory? By studying how the symmetries between the two sides of the duality map into each other one can show that the mass m of a field is related to the scaling dimension Δ of the dual operator [37]. By the dual operator we mean the operator that is sourced by the boundary value of the corresponding field. We will come back to the precise relation between Δ and m in the next section.

4.4.1 Global symmetries in the bulk

The real goal of this thesis is to study superfluids using holography. As we argued earlier, superfluidity can appear if a global symmetry of a quantum field theory gets spontaneously broken. To understand what this might mean in the gravitational side we should understand in more detail how symmetries are realized in the duality. We begin with considering global symmetries in the gravitational side. There are several reasons why such symmetries should not exist, and next we will review how this is seen in asymptotically flat spacetimes. Later we will come back to the $AAdS$ case.

It is a well known fact in string theory that there are no global spacetime symmetries [42]. By spacetime symmetries we mean symmetries of the string theory S-matrix. The S-matrix is calculated from correlation functions on the worldsheet conformal field theory. Thus, the spacetime symmetries appear from symmetries of worldsheet correlation functions. For every such symmetry on the string worldsheet, unless completely accidental, one has a conserved current, which is a dimension one primary field. The conserved current can be used to construct a vertex operator which creates a corresponding gauge boson for each symmetry. Thus, all the symmetries in string theory are really gauge symmetries.

One can argue more generally that in a theory of quantum gravity there should be no global symmetries [43, 44]. The argument is based on black hole physics and goes roughly as follows. If one had a global conserved charge, one could form a black hole by colliding a large number of particles which carry global charge quantum numbers. Due to black hole no-hair theorems, the global charge is not visible outside the black hole, but the black hole looks like a conventional Schwarzschild black hole. Due to the global charge, the information content of the black hole has increased. As is well known, a black hole in asymptotically flat spacetime is not stable, but it radiates out energy in the form of thermal Hawking radiation [45]. Since the radiation is thermal, it cannot carry out macroscopic amounts of the global charge. When the mass of the black hole is of the order of the Planck's mass, Hawking's calculation is expected to break down. Before this happens, one has a situation where the black hole is quite small, but carries an arbitrarily large amount of information content due to the global charges[4]. This is in contradiction with the Bekenstein-Hawking entropy formula which tells us that the black hole should carry an entropy proportional to the area of its horizon.

4.4.2 Local symmetries in the bulk

The next case to consider is local symmetries in the bulk. As an example we can look at rotational symmetry around the S^5. In a theory with dynamical gravity rotational symmetry is a local symmetry. It is a small subgroup of the diffeomorphism group. The conserved charge density from the rotational symmetry is the angular momentum density $L^0_{ab} \propto n_a T_{0b}$ where n^a are 6 coordinates on the S^5 that satisfy $n^a n^a = 1$. One can explicitly see that this rotational symmetry is a gauge symmetry by defining an $SO(6)$ gauge field through the metric as

$$g_{\mu a} \propto A_{\mu ab} n^b. \tag{4.29}$$

The precise full form of the metric can be found in [46]. The field A^{ab}_{μ} is a massless vector field in AdS_5 which carries two $SO(6)$ indices ab, and transforms in the adjoint representation under $SO(6)$ rotations. Now we can see that the angular momentum current L^{μ}_{ab} is a gauge current as it couples to the $SO(6)$ gauge field through the usual gravitational coupling[5]

$$g_{\mu\nu} T^{\mu\nu} \propto A_{\mu ab} n^b T^{\mu a} \propto A_{\mu ab} L^{\mu ab}. \tag{4.30}$$

The $SO(6)$ gauge field is dual to the global symmetry current in the boundary field theory. This is because the boundary value of the field acts as a source to an operator in the boundary field theory. For this to be possible the dual operator must have the same Lorentz and $SO(6)$ indices as the $SO(6)$ gauge field. And furthermore it must be a conserved current, since gauge fields need

[4]This is because the black hole could have been formed from an arbitrarily large number of particles carrying the global charge.

[5]This coupling is correct really when $g_{\mu\nu}$ is a small fluctuation around a backround metric.

to couple to conserved currents. Thus we see that indeed A_μ^{ab} is dual to the $SO(6)_R$ current operator in the boundary field theory.

Conversely, if we have a global symmetry current J_μ in the field theory, there must be a corresponding field in the bulk with a single 4-vector index μ. Let us denote this field by B_μ. On general grounds, a conserved current in a conformal field theory in d spacetime dimensions has a scaling dimension $d-1$. As we mentioned before, scaling dimensions of operators are related to masses of bulk fields. It turns out that scaling dimension $d-1$ corresponds to a massless field [31]. So we can conclude that a global symmetry current in the boundary field theory necessarily leads to a massless particle B_μ in the bulk. On general grounds, such a particle must couple to other fields in a gauge invariant way. Thus, for each global symmetry in the boundary there must be a local symmetry in the bulk.

A byproduct of the previous argument is another viewpoint to the absence of global charges in quantum gravity. Since if there is a global symmetry acting on the fields in the bulk, it produces a global symmetry acting on the operators of the boundary theory. Such a symmetry should have a corresponding conserved current, but now there is no gauge field in the bulk that could provide the dual field in the bulk theory. Thus, there should be no global symmetries in the bulk theory.

Chapter 5

Holographic Superfluids

5.1 Specifying the model

After gathering all the ingredients to describe superfluidity using holography, it is finally time to specify a model we will study. As we argued in the first section, superfluidity arises from spontaneous symmetry breaking of a global $U(1)$ symmetry. In the context of holography, we argued in section 4.4.2 that a global symmetry on the boundary theory must be dual to a local symmetry in the bulk. Thus, a minimal model describing superfluidity must have at least a $U(1)$ gauge symmetry in the bulk. Spontaneous symmetry breaking happens when a scalar operator that transforms under the global symmetry obtains a non-vanishing vacuum expectation value. To have such a scalar operator in the boundary field theory we must have a scalar field Ψ in the bulk, that transforms under bulk $U(1)$ gauge transformations. This is all the necessary ingredients one needs to realize a holographic model of superfluidity. A minimal action with these degrees of freedom is

$$S = \int d^d x \sqrt{-g}\Big(\frac{1}{16\pi G_N}(R-2\Lambda) - \frac{1}{q^2}(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\Psi|^2 + V(|\Psi|^2))\Big) + S_{bdy}. \tag{5.1}$$

Here S_{bdy} is a boundary term whose form we will specify later. The holographic duality we discussed in the last section was between two very specific theories. So one can ask why we might be allowed to choose the gravitational action (5.1). Another way to proceed would be to look for a string theory construction using branes to find a holographic duality between two descriptions for the same system, as Maldacena did for D3 branes. Such a route has been followed eg. in [47–50]. Here we will take another route, which is less rigorous. We will choose the simplest model (5.1) and study that in more detail. In this case there is no guarantee that the gravitational theory will make any sense beyond the classical approximation, and furthermore, the action for the dual field theory is not known. As long as we stick to the classical approximation, the first point will not cause problems. Anyway one can embed the system to string theory if one wants. The second point is not a problem either, in the sense that we can take the gravitational theory to define the dual field theory in terms of the relation (4.25) defining the field theory generating functional. By calculating quantities one can see whether the results obtained this way make sense for a sensible field theory. There have been some steps towards showing[1] that every bulk theory in the classical approximation corresponds to a sensible dual field theory in a large N limit [51]. This makes the author optimistic that (5.1) can indeed define a consistent dual quantum field theory.

In the classical approximation the phases of the dual field theory are determined by $AAdS$ solutions to the equations of motion following from the action (5.1). We will want to study the system at non-vanishing charge density and temperature. The charge density can be obtained by turning on a a chemical potential in the dual field theory. The chemical potential is equivalent to deforming the Hamiltonian of the system as

$$H \to H - \mu Q, \tag{5.2}$$

where Q is the charge operator. As we argued earlier, the boundary value of A_μ couples to the conserved global $U(1)$ current J^μ through the coupling

$$\int d^3x J^\mu A_\mu|_{\partial AdS}. \tag{5.3}$$

Thus, by imposing the boundary conditions $A_t = \mu$ and $A_i = 0$ at the AdS boundary introduces a chemical potential μ in the boundary field theory. This is seen most clearly from (4.25) as

$$\langle e^{-\beta \int d^2\mathbf{x} \mu J^0} \rangle \approx e^{-S_E}, \tag{5.4}$$

where S_E is the Euclidean version of the action (5.1) evaluated on a solution where A_μ satisfies the above boundary conditions.

The finite temperature maps to having periodicity $\beta = 1/T$ in the Euclidean time at the boundary. This follows from the usual definition of the finite temperature equilibrium partition function as a path integral with periodic imaginary time[26]. Thus, we need to find solutions where the spacetime metric is such that it has periodic imaginary time in the AdS boundary. We will approach the problem by starting from high temperature and working our way down to lower temperatures.

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[1] This is in a very specific setting where the bulk theory has a single scalar field and the dual CFT has a single scalar operator $\mathcal{O}$ and nothing else.

5.2 High temperatures

When the temperature of the dual field theory is large compared to the chemical potential, one can approximate the system with a vanishing chemical potential.[2] In this case, the solution corresponding to the thermal state is an AdS-Schwarzschild black hole

$$ds^2 = \frac{L^2}{z^2}(-f(z)dt^2 + \frac{dz^2}{f(z)} + d\mathbf{x}^2), \quad f(z) = 1 - \frac{z^3}{z_H^3}, \quad A_\mu = 0 = \Psi, \quad (5.5)$$

where we have introduced a new more convenient coordinate z related to the earlier AdS coordinate in (4.16) through $z = 1/r$. In particular, the boundary of AdS is at $z = 0$. In (5.5), z_H is related to the Hawking temperature of the black hole, which is identified with the temperature of the dual field theory. The relation can be found in a quick way by requiring that the analytic continuation of the metric to imaginary time is regular near the black hole horizon [28, 52]. We define imaginary time as $\tau = it$ and expand the coordinates as $z = z_H - \rho^2/2$ where ρ is small near the black hole horizon and define an angular coordinate $\theta = \tau|f'(z_H)|/2$. This way the metric becomes

$$ds^2 \approx \frac{2L^2}{|f'(z_H)|^2 z_H^2}(d\rho^2 + \rho^2 d\theta^2). \quad (5.6)$$

To avoid a conical singularity at $\rho = 0$, we have to make the periodic identification $\theta \sim \theta + 2\pi$. In terms of the imaginary time we get

$$\tau \sim \tau + \frac{4\pi}{|f'(z_H)|} = \tau + \frac{4\pi z_H}{3}. \quad (5.7)$$

We see that the AdS-Schwarzschild black hole has a period $\beta = 4\pi z_H/3$ in imaginary time. This corresponds to a temperature

$$T = \frac{3}{4\pi z_H}, \quad (5.8)$$

in the boundary field theory.

Up to this point we simply assumed that $\Psi = 0$ and did not justify it. The basic reason for the vanishing of Ψ is that the coordinate distance from the black hole horizon to the AdS boundary is $z_H \propto 1/T$, which is getting smaller and smaller as the temperature is increased. The field Ψ has to vanish in a power law fashion as $z \to 0$. Since the field has to vary in a shorter z-region the gradient energies grow larger than any potential energies can compensate, so at large temperatures it becomes favorable for the scalar to vanish.

In the large temperature region, the system behaves as a conformal field theory put into finite temperature T. First we can see that the thermodynamics

[2] We give more justification to this procedure later.

of the system is consistent with a CFT. Thermodynamic properties can be determined by going to Euclidean time. To obtain the free energy in the dual field theory we need to evaluate the Euclidean on shell action. The relevant part of the action is

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g}(R - 2\Lambda). \tag{5.9}$$

The Einstein's equations tell us that $R = 4\Lambda$. This way we get the on shell action

$$S = \frac{\Lambda}{8\pi G_N} \int d^3x \int_{\epsilon}^{z_H} \sqrt{-g} = \frac{\Lambda L^4}{24\pi G_N} \Big(\frac{1}{\epsilon^3} - \frac{1}{z_H^3} \Big) \int d^3x. \tag{5.10}$$

Also, we have to take into account two boundary terms. The first one is the Gibbons-Hawking term[3], which has the form

$$S_{GH} = -\frac{1}{8\pi G_N} \int d^3x \sqrt{-h}\Theta, \tag{5.11}$$

where $h_{\mu\nu}$ is the induced metric on the "regularized boundary" at $z = \epsilon$. Θ is the extrinsic curvature of the boundary $\Theta = n^z h^{\mu\nu} \partial_z h_{\mu\nu}$, where n^z is a unit vector perpendicular to the boundary [54]. To cancel off the divergent terms there is also a boundary counter term [55]

$$S_{ct} = -\frac{1}{8\pi G_N} \int d^3x \sqrt{-h} \frac{2}{L}. \tag{5.12}$$

The regularized action now becomes

$$S + S_{GH} + S_{ct} = \frac{4\pi^2 L^2}{3^3 G_N} T^3 \int d^3x. \tag{5.13}$$

Using the basic relation (4.24) we can identify the field theory partition function as

$$Z = \langle 1 \rangle_{CFT} = e^{-S_{grav}(\text{Euclidean})}, \tag{5.14}$$

where the Euclidean action is

$$S_E = -\frac{4\pi^2 L^2}{3^3 G_N} T^2 V_2, \tag{5.15}$$

and V_2 denotes the field theory 2-volume (or area) $V_2 = \int d^2x$. The free-energy is now simply $F = -T \log Z = T S_E$, which gives

$$F = -\frac{4\pi^2 L^2}{3^3 G_N} V_2 T^3. \tag{5.16}$$

[3]This boundary term is required to have a well defined variational principle on a spacetime with a boundary [53].

The field theory energy expectation value is now

$$\langle E\rangle = -\frac{\partial}{\partial\beta}\log Z = \frac{8\pi^2 L^2}{3^3 G_N} V_2 T^3, \tag{5.17}$$

and the entropy is given by

$$\langle S\rangle = -\frac{\partial F}{\partial T} = \frac{4\pi^2 L^2}{9 G_N} V_2 T^2. \tag{5.18}$$

Using the horizon area of the black hole $A = V_2 L^2/z_H^2 = 16\pi^2 L^2 T^2 V_2/9$, which leads to the famous Bekenstein-Hawking entropy

$$\langle S\rangle = \frac{A}{4G_N}. \tag{5.19}$$

We see the thermodynamic quantities are indeed those of a $2+1$ dimensional conformal field theory. Another important thing is that the free energy is proportional to L^2/G_N, which is supposed to be very large for the classical gravity approximation to be applicable. Usually when the free-energy of a system is of the form

$$F = cT^d, \tag{5.20}$$

one can interpret c as being proportional to the degeneracy of states at a given energy. Thus, we might want to identify L^2/G_N as the degeneracy of states in the boundary field theory. In this sense it seems like there is a large degeneracy of states in the boundary field theory. It is suggestive that there is a large symmetry rotating the degenerate states to each other. Such a symmetry ought to be accompanied with conserved currents. But as we do not have a large number of gauge fields in the bulk, we would arrive in a contradiction. This contradiction can be avoided by identifying the large symmetry as a gauge symmetry. Then there is no need to have a corresponding dual field in the bulk, since a gauge theory current operator is not gauge invariant and thus, not a physical observable. This suggests that the boundary field theory is a large N gauge theory even in our case.

If one is interested in the low energy dynamics of the finite temperature system, one can perturb the AdS-Schwarzschild black hole slightly out of equilibrium. Then one can solve Einstein's equations in a derivative expansion in the field theory directions [56]. This leads to equations that are equivalent to relativistic hydrodynamics of a 2+1 dimensional fluid with a shear viscosity $\eta = s/4\pi$, where s is the entropy density of the fluid [56]. The smallness of the viscosity to entropy density ratio suggests that the dual field theory is strongly interacting[4].

[4] A rather naive argument for the strong interactions is that in a quasiparticle picture of a fluid $\eta \propto 1/\sigma$, where σ is a typical quasiparticle scattering cross-section [57].

The viscosity of the dual field theory fluid may be shown to be equivalent to a viscosity that can be assigned to the surface of a black hole through the so called membrane paradigm [58]. Furthermore it can be shown that η/s depends only on the form of the gravitational action, and not on the details of the black hole solution. Thus, the viscosity associated with the total motion of energy-momentum is independent of the phase of the system.

5.3 Intermediate temperatures

As we lower the temperature, μ/T is no longer negligible. The system starts to look like it has an overall global charge. We will assume that Ψ has not yet condensed. As the chemical potential is identified with the boundary value of A_t, we will have a black hole solution with a non-vanishing electric field. This black hole solution is the AdS-Reissner-Nordström (AdS-RN) black hole

$$ds^2 = \frac{L^2}{z^2}\Big(- f(z)dt^2 + \frac{dz^2}{f(z)} + d\mathbf{x}^2\Big), \quad A_t(z) = \mu(1 - z/z_H), \tag{5.21}$$

$$f(z) = 1 - \Big(1 + \frac{z_H^2\mu^2}{\gamma^2}\Big)\Big(\frac{z}{z_H}\Big)^3 + \frac{z_H^2\mu^2}{\gamma^2}\Big(\frac{z}{z_H}\Big)^4, \quad \gamma^2 = \frac{L^2 q^2}{4\pi G_N}. \tag{5.22}$$

Now we see that when T/μ is large, this reduces back to the AdS-Schwarzschild black hole as $z_H^2\mu^2 \to 0$. Due to scaling symmetries of the equations of motion following from the action (5.1) the only physically relevant parameter is the dimensionless ratio T/μ, while T dependence can be worked out by using dimensional analysis. Again we can work out the thermodynamic quantities of the system as follows. The relevant part of the action is now

$$S = \int d^4x\sqrt{-g}\Big(\frac{1}{16\pi G_N}(R - 2\Lambda) - \frac{1}{4q^2}F_{\mu\nu}F^{\mu\nu}\Big). \tag{5.23}$$

Using the equations of motion we again obtain $R = 4\Lambda$ and using $F_{z0} = -\mu/z_H$ we get

$$S = \int d^4x\sqrt{-g}\Big(\frac{\Lambda}{8\pi G_N} - \frac{\mu^2}{2q^2 z_H^2}g^{zz}g^{00}\Big). \tag{5.24}$$

Again this is to be integrated over the z direction. After introducing the boundary terms we used in the neutral black hole, we get

$$S + S_{bdy} = \Big(\frac{L^2}{16\pi G_N}\frac{1}{z_H^3} + \frac{\mu^2}{2q^2 z_H}\Big)\int d^3x. \tag{5.25}$$

The identification of the temperature goes as before, except $f'(z_H)$ is different for the charged black hole, and we obtain the temperature

$$T = \frac{|f'(z_H)|}{4\pi} = \frac{1}{4\pi z_H}\Big(3 - \frac{z_H^2\mu^2}{\gamma^2}\Big). \tag{5.26}$$

Again we can analytically continue to Euclidean time to give the grand canonical partition function [59]

$$\Omega = -\frac{L^2}{16\pi G_N} V_2 \frac{1}{z_H^3} - \frac{1}{2q^2} V_2 \mu^2 \frac{1}{z_H}. \tag{5.27}$$

This time it is a bit trickier to write Ω as a function of the temperature. The first part of the expression is identical to the high temperature result while the second part is the contribution from degrees of freedom carrying the global charge. The classical approximation requires that $1/q^2$ is large, which tells us that the number of charged degrees of freedom is large, and proportional to $1/q^2$.

For small μ/T we can solve (5.26) in a power series and substitute the result to the grand canonical partition function to give

$$\Omega = -\frac{V_2 L^2}{G_N} \Big(\frac{4\pi^2}{27} T^3 + \frac{1}{4\gamma^2} T\mu^2 + O(\mu^4/T) \Big). \tag{5.28}$$

which is of the form $\Omega = -V_2 T^3 g(T/\mu)$, for some function g, as expected from conformal symmetry.

Again one can study the system out of equilibrium. This leads to hydrodynamics with a conserved current. As emphasized earlier, the viscosity has a universal value $\eta = s/4\pi$. There is a new response coefficient, which is the conductivity in the field theory, that can be shown to take the value $\sigma = 1/q^2$ [58]. So the horizon behaves as a fluid with finite charge density and temperature.

5.4 Low temperatures

As one further lowers the temperature of the system, the scalar field starts to play an important role. Let us for a moment pretend that the scalar field is not there in the theory. In this case, as one takes the temperature to zero, the *AdS*-RN black hole becomes extremal. In this case, the two first order poles in the metric function (5.22) approach each other as μ/T is increased and eventually they collide to form a second order pole. This means that the black hole temperature is zero, while it has a finite horizon radius z_H. By the Bekenstein-Hawking formula this means that it has a finite entropy, which can be also calculated from (5.27) to be

$$\langle S \rangle = \frac{\pi}{3q^2} V_2 \mu^2. \tag{5.29}$$

This is problematic for the boundary field theory as it contradicts the third law of thermodynamics, which states that a zero temperature system should have a vanishing entropy. One might guess that even a slight disturbance of the system would destroy the degeneracy and pick a unique ground state[5]. Such studies have indeed been performed by subjecting the system to an external magnetic field [60] and the zero temperature limit seems to go to zero entropy.[6]

What saves the situation in our case is that the scalar field condenses and completely changes the zero temperature limit. The physics of the system is no longer as universal as it was for the high temperature systems. This is because the high temperature properties only depended on the properties of the black hole. As we will soon see, the scalar field will develop a non-vanishing profile outside the black hole, whose properties will depend on the choice of the potential $V(\Psi)$. This forces us to give up the universality of the physics and choose a form for $V(\Psi)$. In the rest of this thesis we will work with the minimal possible choice

$$V(\Psi) = m^2|\Psi|^2. \tag{5.30}$$

Furthermore, we will choose $m^2 = -2/L^2$ for the explicit numerical calculations. At first sight it seems suprising that it is at all possible to have a negative m^2. What saves the situation is that the scalar field must vanish as a power-law as one approaches the AdS boundary and unless m^2 is below the Breitenlohner-Freedman bound $m^2 \geq -9/(4L^2)$ [62], the gradient energies will win over the negative potential energy that can be gained by rolling down the potential to $|\Psi| \to \infty$.

Now we can come back to the question what happens when the temperature is lowered. This is equivalent to increasing the chemical potential since, as explained earlier, the only relevant parameter for the physics is the dimensionless ratio μ/T. One can see from the action (5.1), that the backround value of the gauge field $A_t = \mu(1 - z/z_H)$ acts as an effective mass term for the scalar field

$$m_{eff}^2 = m^2 + g^{tt}A_t^2. \tag{5.31}$$

So increasing the chemical potential makes the mass of the scalar field more and more tachyonic since $g^{tt} \leq 0$. By studying fluctuations of the scalar field around the *AdS*-RN black hole one finds that the solution with a vanishing scalar field is unstable when m^2 is sufficiently small and μ/T is increased [63].

To find the correct thermal state of the system, we look for other solutions to the equations of motion following from the action (5.1). The solution which has the lowest free energy determines the correct thermal state of the system.

As a final choice of parameters we will choose q^2 to be large compared to G_N/L^2. This choice is done purely for calculational reasons as it allows us to neglect the backreaction of the scalar and the gauge field to the metric.

[5] Unless there is supersymmetry that protects the degeneracy.

[6] It has been also argued that the extremal black hole has zero entropy [61]. This would mean that the Bekenstein-Hawking formula does not apply in that case and that the zero temperature limit of the *AdS*-RN solution is discontinuous [61]. This does not solve our problem since still the entropy at an arbitrarily small temperature is macroscopically non-vanishing [59].

This follows because the matter energy-momentum tensor sourcing Einstein's equation is proportional to $G_N/L^2q^2 \ll 1$, so we can approximate the matter energy-momentum tensor as being zero. To see what such an approximation means in terms of the boundary field theory we recall from (5.27) that $1/q^2$ is proportional to the number of charged degrees of freedom, while L^2/G_N is proportional to the overall number of degrees of freedom. The choice of parameters then means that we are setting the number of charged degrees of freedom in the boundary field theory to be lot smaller than the number of overall degrees of freedom.

In this approximation the relevant equations of motion we need to solve are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{5.32}$$

$$\frac{1}{\sqrt{-g}}D_\mu(\sqrt{-g}g^{\mu\nu}D_\nu\Psi) + m^2\Psi = 0, \tag{5.33}$$

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{-g}F^{\mu\nu}) = i(\Psi^*\partial^\nu\Psi - \Psi\partial^\nu\Psi^* - 2iA^\nu|\Psi|^2). \tag{5.34}$$

The solution to the Einstein's equations (5.32) is simply the AdS-Schwarzschild black hole. To solve rest of the equations (5.33) and (5.34) we need to make some simplifying assumptions. We will assume that the solution minimizing the Euclidean on-shell action is translationally invariant in the $\mathbf{x}$-directions and independent of time. Thus, we can set $A_x = A_y = 0$ up to pure gauge configurations. As a gauge fixing condition we set $A_z = 0$ [64]. Now the Maxwell's equations (5.34) force the phase of Ψ to be constant. Thus, we are left with an ansatz $\Psi(z) = R(z)/\sqrt{2}$ and $A_t(z) = A(z)$, where both $R(z)$ and $A(z)$ are real. Substituting this ansatz into equations (5.33) and (5.34) we obtain

$$z^4\partial_z\Big(\frac{f(z)}{z^2}\partial_z R(z)\Big) + \Big(m^2L^2 - \frac{A(z)^2z^2}{f(z)}\Big)R(z), \tag{5.35}$$

$$f(z)\partial_z^2 A(z) + \frac{R(z)^2}{z^2}A(z) = 0. \tag{5.36}$$

Before we can solve these equations we have to specify boundary conditions. At the black hole horizon we will require regularity. For the gauge field we require that $A_\mu A^\mu = g^{tt}A_t^2 = z^2A(z)^2/f(z)$ is finite at the horizon, which leads to $A(z = z_H) = 0$. Furthermore we assume that the scalar field is finite at the horizon, which together with (5.35) leads to a regularity condition

$$f'(z_H)\partial_z R(z_H) + \frac{m^2L^2}{z_H^2}R(z_H) = 0. \tag{5.37}$$

Next we need to impose boundary conditions at the AdS boundary $z = 0$. Equation (5.35) implies that near $z = 0$, $R(z)$ behaves as

$$R(z) = R_-z^{\Delta_-} + R_+z^{\Delta_+} + ..., \tag{5.38}$$

where

$$\Delta_{\pm} = \frac{3}{2} \pm \sqrt{\frac{9}{4} + L^2 m^2}. \tag{5.39}$$

This asymptotic behavior holds even more generally as

$$\Psi(x, z) = \Psi_-(x) z^{\Delta_-} + \Psi_+(x) z^{\Delta_+}. \tag{5.40}$$

First, let us identify Ψ_- as the boundary value of Ψ which is to be interpreted as a source for a scalar operator $\mathcal{O}_-$ in the dual field theory. The expectation value of the dual operator can be calculated by using equation (4.24) as

$$\langle \mathcal{O}_-(x) \rangle = \frac{\delta S}{\delta \Psi_-^*(x)} = \frac{1}{q^2} \lim_{z \to 0} \int d^3x' \sqrt{-g} g^{zz} \frac{\delta \Psi^*(x', z)}{\delta \Psi_-^*(x)} \partial_z \Psi(x', z), \tag{5.41}$$

where have assumed Ψ satisfies its equations of motion, and integrated by parts. Substituting in the expansion (5.40) gives

$$\langle \mathcal{O}_-(x) \rangle = \frac{\Delta_+}{q^2} \Psi_+(x), \tag{5.42}$$

where $\Psi_+ = R_+/\sqrt{2}$ for our solution. Thus, when interpreting R_- as the source, the operator expectation value will be given by the subleading term R_+. So to calculate the one point function we want to set the source R_- to zero. This gives us a second boundary condition to solve (5.35).

At this point we can obtain the advertised connection between operator scaling dimensions and the masses of bulk fields, for scalar operators. We recall that a scale transformation in the boundary field theory maps to the transformation $(t, \mathbf{x}, z) \to \lambda^{-1}(t, \mathbf{x}, z)$ in the bulk. This can be thought of as a change of variables. The scalar field does not transform under such a transformation (since it is a scalar). Thus, the coefficient Ψ_+ must scale as $\Psi_+ \to \lambda^{\Delta_+} \Psi_+$. This means that the operator $\mathcal{O}$ must scale under a boundary field theory scale transformation as

$$\mathcal{O} \to \lambda^{\Delta_+} \mathcal{O}. \tag{5.43}$$

This is what one requires for an operator with a scaling dimension Δ_+. Thus, we learn that the scaling dimension of the boundary theory operator is $3/2 + \sqrt{9/4 + m^2 L^2}$.

It should be mentioned that it is also possible to treat Ψ_+ as the source in the range of the scalar field masses [65]

$$-\frac{9}{4} \leq m^2 L^2 < \frac{5}{4}. \tag{5.44}$$

In this case the operator expectation value is proportional to Ψ_- and the corresponding operator has a scaling dimension Δ_-. We refer to [65] for more details.

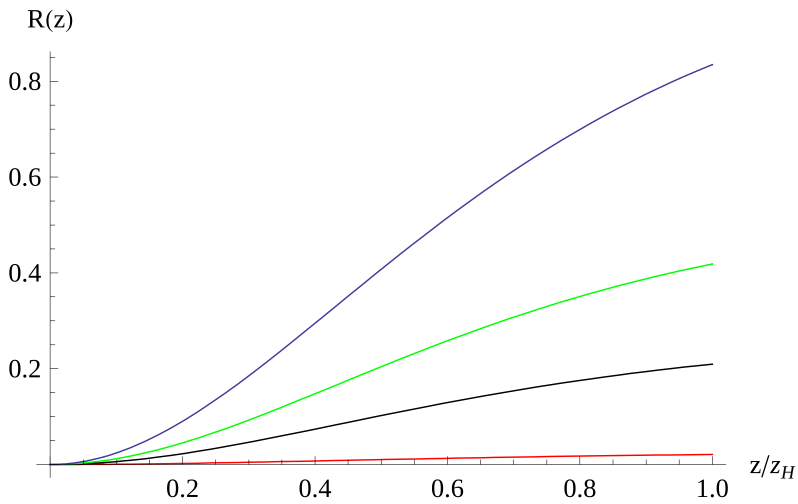

Figure 5.1: Scalar field profiles for different values of $T/|\mu|$. From bottom to up, the corresponding values are $T/|\mu| = (0.24607, 0.24601, 0.24583, 0.24508)$.

Now we are ready to solve the equations (5.35) and (5.36). Numerical solutions with $m^2 = -2/L^2$ for different values of T/μ are shown in Figure 5.1.

When T/μ is below a certain critical value, one finds solutions with a non-vanishing profile for the scalar field, or in other words, the scalar field condenses. To really see that the condensed state is thermodynamically preferred one can calculate the value of the on-shell action

$$S = \frac{1}{q^2}\int d^4x\sqrt{-g}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |D_\mu\Psi|^2 - m^2|\Psi|^2\Big)$$
$$= -\frac{1}{2q^2}\int d^4x\sqrt{-g}\Big(\frac{1}{\sqrt{-g}}\partial_z(\sqrt{-g}A_\nu F^{z\nu}) + \frac{1}{\sqrt{-g}}\partial_z(\sqrt{-g}g^{zz}R\partial_z R) - A_\mu A^\mu R^2\Big).$$

We would like to compare the action between the solutions with and without the condensate in the same chemical potential and temperature. Substracting the two actions[7] gives the difference

$$\Delta S = -\frac{1}{2q^2}\int d^3x\Big(A(0)(A'(0) + A(0)) + \int dz\frac{R(z)^2A(z)^2}{z^2f(z)}\Big). \tag{5.45}$$

Next we go to the Euclidean signature and identify the free-energy difference as

$$\Delta\Omega = -T\Delta S_E = \frac{1}{2q^2}V_2\Big(A(0)(A'(0) + A(0)) + \int dz\frac{R(z)^2A(z)^2}{z^2f(z)}\Big). \tag{5.46}$$

[7] Using the fact that for the solution with vanishing scalar field $A'(0) = -A(0)$.

If $\Delta\Omega < 0$, the solution with a non-vanishing scalar profile is thermodynamically favored. Again we can use a convenient parametrization of the free-energy

$$\Delta\Omega = V_2 T^3 \frac{g(T/\mu)}{q^2}. \tag{5.47}$$

By numerically evaluating $g(T/\mu)$ for solutions at different temperatures we obtain Figure 5.2. From Figure 5.2 we see that indeed when the temperature is below a critical temperature $T_c \approx 4.065\mu$ the solution with a non-vanishing scalar profile is thermodynamically favored.

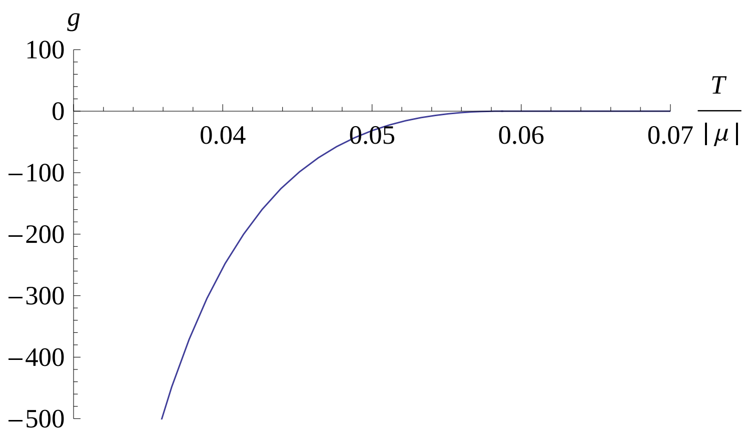

Figure 5.2: The free-energy as a function of $T/|\mu|$.

From Figure 5.2 one can also see that the phase transition to the condensed phase is of second order. This provides an interesting example of how a second order phase transition can be realized in holography.

From the solutions we can easily calculate the value of the condensate in the dual field theory by using (5.42). The result of the numerical calculation is shown in Figure 3. Near the phase transition we see a standard mean field scaling

$$\langle \mathcal{O} \rangle \propto \sqrt{1 - \frac{T}{T_c}}. \tag{5.48}$$

In fact one finds the same mean field scaling for many values of the scalar mass m^2 studied in [66].

5.5 The Goldstone mode

What is the physical interpretation of the appearance of a non-vanishing profile for the Ψ field? Since Ψ transforms under the local bulk $U(1)$ symmetry

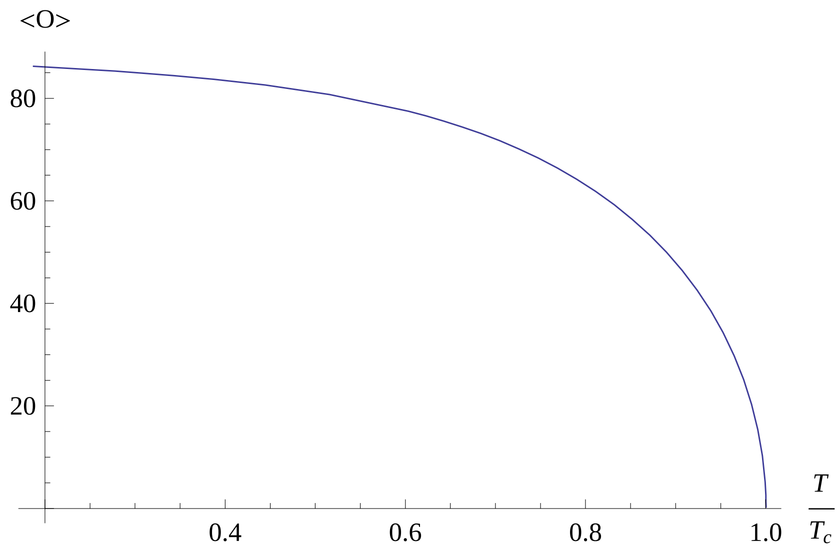

Figure 5.3: The value of the condensate as a function of T/T_c.

$\Psi \to e^{i\Lambda}\Psi$, $A_\mu \to A_\mu - \partial_\mu \Lambda$, one can interpret the non-vanishing profile of Ψ as spontaneous breaking of this symmetry. On the other hand there is a non-vanishing expectation value $\langle \mathcal{O} \rangle$ in the dual field theory, which transforms under the global $U(1)$ symmetry. Thus, the global symmetry is spontaneously broken below T_c. For this picture to be consistent, the Higgs mechanism in the bulk has to be dual to spontaneous symmetry breaking of a global symmetry in the boundary field theory. So somehow there must exist a Goldstone mode in the dual field theory.

Indeed one sees the Goldstone mode in correlation functions in the broken phase [67, 68]. From the correlation functions one finds a pole with a linear dispersion relation $\epsilon_k = \pm c_s |\mathbf{k}|$, which can be identified as the Goldstone mode. Also one finds from the correlation functions of [67] that the other modes are gapped, so that Landau's criterion is satisfied.

We can construct a low energy effective action for the Goldstone mode in the zero temperature limit by using symmetry arguments, similar to [69]. The key point is that the Goldstone mode $\chi(x, z = 0) = \phi(x)$ enters through gauge symmetry in the combination $B_\mu = \partial_\mu \phi + a_\mu$, where $a_\mu(x) = A_\mu(x, z = 0)$ is the dual field theory source. As in [69] one can define a quantum effective action Γ for the Goldstone field ϕ, by integrating out the modulus of the order parameter in the usual 1PI quantum effective action. Because of the above gauge symmetry, Γ depends only on the combination B_μ. It can be expanded for small values of B_i and $B_t - \mu$ to give an effective action, as

$$S_{eff} = \Gamma[B_\mu] = \int d^3x \Big(\frac{1}{2}\kappa_t (B_t - \mu)^2 - \frac{1}{2}\kappa_x B_i^2 + ...\Big). \qquad (5.49)$$

The reason the expansion starts from second order is that the first order terms are either constants independent of ϕ or total derivatives of ϕ, which do not affect the dynamics of that mode. We can determine κ_t and κ_x by looking at the "vacuum" where $\phi = 0$, $a_0 = \mu$ and $a_i = 0$. In this case the quantum effective action reduces to the logarithm of the usual partition function $\Gamma = -i \log Z$ and we can use the statistical physics identity $-i \log Z = \int d^3x P$, where P is the pressure. This leads to

$$\kappa_t = \frac{\partial^2 P}{\partial \mu^2}, \quad \kappa_x = -\frac{\partial^2 P}{\partial a_x^2}|_{a_x=0}. \qquad (5.50)$$

These quantities may be calculated numerically from solutions with superfluid flows as is described in [70, 71] and the resulting dispersion relation indeed agrees with those extracted from the correlation function in [67, 68] even at non-vanishing temperatures. The Goldstone effective action is thus

$$S_{eff} = \int d^3x \Big(\frac{1}{2}\kappa_t (\partial_t \phi)^2 - \frac{1}{2}\kappa_x (\partial_x \phi)^2 + ...\Big). \qquad (5.51)$$

A reader familiar with the Coleman-Mermin-Wagner theorem [72, 73] might be puzzled that there is a finite temperature spontaneous symmetry breaking of a continuous symmetry in 2+1 dimensions. The reason we see spontaneous symmetry breaking, is the small q^2 limit we are studying. At one loop level in the bulk, one finds that the phase of the condensate is indeed randomized [68] and long range order is transformed into algebraic long range order.[8] This is what one anticipated from the dual field theory side and was confirmed in the gravitational side in [68]. This phenomenon is more surprising in the gravitational side, where one did not expect to see such strong infrared fluctuations as the bulk theory is $3 + 1$ dimensional. This was interpreted in [68] as being due to the existence of a horizon. This is a highly non-trivial check that the 3+1 dimensional quantum gravitational system behaves as a 2+1 dimensional quantum field theory.

[8]Meaning that the condensate two point function does not approach a constant, but decays as a power law as the operators are separated.

Chapter 6

Discussion

The bulk description provides a nice picture of the two fluid model, which we discussed earlier in section (3.4). There is a condensate of charged scalar Ψ particles outside the black hole, with wavefunction $R(z)/\sqrt{2}$. This is the superfluid part of the fluid. Then there is the horizon of the black hole, that behaves as a charged normal fluid with its associated viscosities and conductivities [58]. This is the normal part of the fluid. This setup provides us with a simple model where the interactions between the two fluids are described in a simple unified way. In this model, the finite temperature fluctuations are all encoded in the classical black hole solution.

Such a unified description of the two fluids becomes especially powerful when compared to traditional condensed matter methods when one is studying solitons. To study solitons at finite temperature T using traditional condensed matter methods one should in principle solve for the energy eigenvalues of a many-body Schrödinger equation (with possibly 10^6 particles to model trapped atomic superfluids) up to energies $E > T$ to obtain the thermal partition function. This is not a particularly easy task. In practice, one often uses a mean field approximation of some kind to model the solitons [16]. So alternative methods, such as holography, are certainly welcome in studying finite temperature solitons. As an introduction, this sets the stage for the main subject of this thesis, which is the study of superfluid solitons using holographic methods.

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